ADAPTIVE AND DYNAMICALLY CONSTRAINED PROCESS NOISE ESTIMATION FOR ORBIT DETERMINATION

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This paper introduces two new algorithms to accurately estimate Kalman filter process noise online for robust orbit determination in the presence of dynamics model uncertainties. Common orbit determination process noise techniques, such as state noise compensation and dynamic model compensation, require offline tuning and a priori knowledge of the dynamical environment. Alternatively, the discrete time process noise covariance can be estimated through adaptive filtering. However, current adaptive filtering techniques often use ad hoc methods to ensure the estimated process noise covariance is positive semi-definite and cannot accurately extrapolate over measurement outages. Furthermore, adaptive filtering techniques do not constrain the discrete time process noise covariance according to the underlying continuous time dynamical model, and there has been limited work on adaptive filtering with colored process noise. To overcome these limitations, a novel approach is developed which optimally fuses state noise compensation and dynamic model compensation with covariance matching adaptive filtering. The adaptability of the proposed algorithms is a significant advantage over state noise compensation and dynamic model compensation. In contrast to existing adaptive filtering approaches, the new techniques are able to accurately extrapolate over gaps in measurements. Additionally, the proposed algorithms are more accurate and robust than covariance matching, which is demonstrated through two case studies: an illustrative example and two spacecraft orbiting an asteroid.

INTRODUCTION

In orbit determination, there are always differences between the modeled and true spacecraft dynamics due to complex forces which cannot be modeled perfectly. These forces include gravity, solar radiation pressure, atmospheric drag, third-body perturbations, tidal effects, and propulsive maneuvers. Furthermore, reduced order dynamics models are often used for onboard navigation due to computational limits. In Kalman filtering, dynamics modeling deficiencies are known as process noise. Inaccurate process noise models can lead to large estimation errors as well as filter inconsistency and divergence.^{1,2} Modeling process noise for asteroid missions is especially challenging because the dynamical environment is poorly known a priori, and the process noise can change significantly as the spacecraft transitions between high and low altitude orbits. It is increasingly difficult when there is limited human intervention such as in the Autonomous Nanosatellite Swarming (ANS)^{3–5} mission concept, which utilizes an autonomous swarm of small spacecraft to characterize an asteroid. This paper presents two adaptive and dynamically constrained process noise estimation algorithms that are applicable even when the dynamical environment is not known a priori and the process noise is time-varying.

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Two common approaches for modeling process noise in orbit determination are state noise compensation (SNC) and dynamic model compensation (DMC).^{6–14} These methods explicitly attribute spacecraft state process noise to unmodeled accelerations, which is the difference between the modeled and true accelerations. SNC treats the unmodeled accelerations as continuous time (CT) process noise, which is assumed to be a zero-mean white Gaussian process with known covariance. However, in reality unmodeled accelerations are correlated in time. DMC allows for time correlation of the unmodeled accelerations by augmenting the state with empirical accelerations. The dynamics of the empirical accelerations are frequently modeled as a first-order Gauss-Markov process that treats the unmodeled accelerations allows DMC to provide higher orbit determination accuracy than SNC.^{12, 13, 15} DMC also provides a direct estimate of the unmodeled accelerations, which may be desirable depending on the application.^{6, 13} On the other hand, SNC is simpler to implement and less computationally expensive than DMC.

A major drawback of SNC and DMC is that time-intensive offline tuning is required to determine the CT process noise covariance.^{6, 14–16} The tuned covariance is no longer optimal when the process noise changes, which naturally occurs over time due to variations in space weather and spacecraft properties. The process noise also changes whenever the orbit is altered as well as between periapsis and apoapsis for eccentric orbits. Furthermore, SNC and DMC are poorly suited to scenarios where the dynamical environment is not well known a priori because the offline tuning cannot be accurately completed before the mission. Genetic model compensation (GMC) was developed to adaptively tune the main diagonal of the CT process noise covariance for DMC through a genetic optimization algorithm.^{16, 17} However, GMC is not widely used due to its complicated implementation and sensitivity to numerous hyperparameters.

Alternatively, the discrete time (DT) process noise covariance may be estimated through adaptive filtering techniques, which are commonly divided into four categories: Bayesian,^{18–22} maximum likelihood,^{23,24} correlation,^{25,26} and covariance matching (CM).^{27–29} An early survey on adaptive filtering techniques is given by Mehra.² A more recent and comprehensive survey is provided by Duník et al.³⁰ with a focus on correlation methods. The various adaptive filtering approaches are derived under different assumptions and have diverse limitations. For example, some techniques are only applicable to linear time invariant systems.^{25,28} Often, ad hoc methods are used to ensure the DT process noise covariance is positive semi-definite (PSD). These ad hoc methods include truncating terms,^{31,32} setting each diagonal element of the process noise covariance equal to its absolute value,^{29,33} finding the nearest positive semi-definite matrix in the Frobenius norm sense,^{7,32,34} and using the previous estimate of the DT process noise covariance if the current estimate is not PSD.²⁸ Adaptive filtering approaches also vary significantly in their computational cost. CM techniques are widely used because they are computationally efficient and simple to implement. Since many other adaptive filtering algorithms are not computationally tractable for typical spacecraft onboard processors, several authors have explored the use of CM for orbit determination.^{7,27,29,32}

However, there are limitations to directly applying current adaptive filtering techniques to orbit determination. For instance, ad hoc methods used in adaptive filtering to ensure the DT process noise covariance is PSD are not necessarily appropriate for orbit determination and can lead to biased estimates of the DT process noise covariance. Adaptive filtering techniques also have a limited ability to extrapolate over measurement outages where the DT process noise covariance may be significantly different from previous values. Additionally, existing adaptive filtering approaches estimate the DT process noise covariance directly without constraining the estimate according to the

underlying CT dynamical model. Consequently, the estimated DT process noise covariance may not be realizable given that the process noise is ascribed to CT unmodeled accelerations. Finally, in orbit determination the process noise is often highly correlated in time. However, there has been limited work on adaptive filtering for systems with colored process noise,³⁵ which Duník et al.³⁰ recently identified as a necessary area of future research in adaptive filtering.

This paper overcomes the aforementioned limitations of SNC, DMC, and adaptive filtering by optimally fusing SNC and DMC with CM through a constrained weighted least squares optimization. This yields two novel, adaptive, and dynamically constrained process noise techniques called adaptive SNC (ASNC) and adaptive DMC (ADMC). The adaptability of the proposed algorithms is a significant advantage over SNC and DMC. Additionally, the developed techniques are more accurate than CM because they are dynamically constrained and avoid common ad hoc methods for ensuring the DT process noise covariance is PSD. In particular, ADMC can provide more accurate estimation than CM and ASNC for systems with colored process noise. In addition to Earth-based missions, ASNC and ADMC are well suited to asteroid missions such as ANS^{3–5} where the dynamical environment is poorly known a priori, the process noise is time-varying, and there is limited human intervention. Another potential use for ASNC and ADMC is to tune the process noise covariance in advance for missions that will use either SNC or DMC.

The following section provides background on SNC, DMC, and CM. The next section describes how SNC and DMC are each optimally fused with CM through a constrained weighted least squares minimization to yield ASNC and ADMC. This includes a derivation of the weighting matrix used in the least squares minimization as well as discussion on efficiently solving the optimization problem. In the subsequent section, the advantages of ASNC and ADMC are demonstrated through an illustrative example of a particle moving in one dimension. The benefits of the developed techniques are further substantiated in the penultimate section through a more realistic and challenging scenario of two spacecraft orbiting an asteroid. Finally, conclusions are presented based on the results of the two case studies, and future work is outlined.

BACKGROUND

State Noise Compensation

In orbit determination, the dynamical model is often linearized such that the state dynamics can be represented as a linear time-varying system subject to process noise described by⁶

$$\dot{\boldsymbol{x}}(t) = \mathbf{A}(t)\boldsymbol{x}(t) + \boldsymbol{\Gamma}(t)\boldsymbol{\epsilon}(t)$$
(1)

Here A is the plant matrix, Γ is the process noise mapping matrix, ϵ is the CT process noise, and x is the system state, which contains either a Cartesian or orbital element spacecraft state. For a linear system, the only sources of spacecraft state process noise are unmodeled accelerations and numerical error. In SNC, the process noise is attributed entirely to unmodeled accelerations since they generally create orders of magnitude more process noise than numerical error. The unmodeled accelerations are described by ϵ , which is modeled as a zero-mean white Gaussian process.^{6,33} The realization of each ϵ is an acceleration, and the covariance of ϵ is given by

$$\mathbf{E}[\boldsymbol{\epsilon}(t)\boldsymbol{\epsilon}(\tau)^{T}] = \widetilde{\mathbf{Q}}(t)\delta(t-\tau)$$
(2)

where $\delta(\cdot)$ is the Dirac delta function. The solution to Eqn. (1) can be written as

$$\boldsymbol{x_k} = \boldsymbol{\Phi_k} \boldsymbol{x_{k-1}} + \boldsymbol{w_k} \tag{3}$$

where x_k is the state at time step k, $\Phi_k = \Phi(t_k, t_{k-1})$ is the state transition matrix (STM) that propagates the state over the measurement update interval from time t_{k-1} to time t_k , and

$$\boldsymbol{w}_{\boldsymbol{k}} = \int_{t_{k-1}}^{t_k} \boldsymbol{\Phi}(t_k, \tau) \boldsymbol{\Gamma}(\tau) \boldsymbol{\epsilon}(\tau) d\tau$$
(4)

is the DT process noise. It is easily shown that w is uncorrelated in time and that $w_k \sim \mathcal{N}(0, \mathbf{Q}_k)$ where the DT process noise covariance is^{6,29}

$$\mathbf{Q}_{k} = \int_{t_{k-1}}^{t_{k}} \mathbf{\Phi}(t_{k},\tau) \mathbf{\Gamma}(\tau) \widetilde{\mathbf{Q}}(\tau) \mathbf{\Gamma}(\tau)^{T} \mathbf{\Phi}(t_{k},\tau)^{T} d\tau$$
(5)

In SNC, $\widetilde{\mathbf{Q}}$ is assumed constant and is tuned offline. To facilitate tuning, the elements of ϵ are generally assumed to be independent such that $\widetilde{\mathbf{Q}}$ is diagonal.^{6,15,29} The DT process noise covariance is used in the Kalman filter time update of the formal covariance which is given by

$$\mathbf{P}_{k|k-1} = \mathbf{\Phi}_{k} \mathbf{P}_{k-1|k-1} \mathbf{\Phi}_{k}^{T} + \mathbf{Q}_{k}$$
(6)

Here, $\mathbf{P}_{k|k-1}$ is the time updated formal covariance at time step k, and $\mathbf{P}_{k-1|k-1}$ is the measurement updated formal covariance at time step k-1.

Dynamic Model Compensation

In reality, the unmodeled accelerations are correlated in time. DMC takes this into account by augmenting the state vector with empirical accelerations, which are also referred to in literature as ficticious or compensative accelerations.¹⁶ Although higher order models may be used,¹⁴ empirical accelerations are often modeled as a first-order Gauss-Markov process given by^{6,12}

$$\dot{\boldsymbol{a}}_{\boldsymbol{e}}(t) = -\beta \boldsymbol{a}_{\boldsymbol{e}}(t) + \boldsymbol{\epsilon}(t) \tag{7}$$

which is a linear stochastic differential equation known as a Langevin equation.¹⁶ Here a_e is a vector containing three orthogonal empirical accelerations. Again, ϵ is a zero-mean white Gaussian process whose covariance is $\tilde{\mathbf{Q}}$. The matrix $\tilde{\mathbf{Q}}$ is assumed constant, and both $\tilde{\mathbf{Q}}$ and β are determined through offline tuning. To facilitate tuning, it is typically assumed that the empirical accelerations are independent from one another such that $\tilde{\mathbf{Q}}$ and β are diagonal.^{12, 13, 15, 16} The offline tuning can be performed manually through trial and error¹⁵ or by fitting the empirical acceleration autocovariance model to sample accelerations.¹⁴ Alternatively, β can be estimated as part of the state. However, when estimating β the performance of DMC is sensitive to the modeled process noise covariance of β , which is tuned offline.¹³ After determining $\tilde{\mathbf{Q}}$ and β , the DT process noise covariance matrix as shown in Eqn. (6). Note that Φ and Γ are different for DMC and SNC since DMC augments the state with empirical accelerations.

Under the previously stated assumptions, the solution to Eqn. (7) is⁶

$$a_{e_i}(t) = e^{-\beta_i(t-t_0)} a_{e_i}(t_0) + \int_{t_0}^t e^{-\beta_i(t-\tau)} \epsilon_i(\tau) d\tau$$
(8)

where a_{e_i} is the *i*th component of a_e , ϵ_i is the *i*th component of ϵ , and β_i is the *i*th element of the main diagonal of β . The first term in Eqn. (8) is deterministic and is included in the filter dynamics

model. The second term is stochastic with a mean of zero and is taken into account through the DT process noise covariance. The autocorrelation function of each empirical acceleration is given by⁶

$$\mathbf{E}[a_{e_i}(t)a_{e_i}(\tau)] = \Psi(\tau,\tau)e^{-\beta_i(t-\tau)}$$
(9)

where

$$\Psi(\tau,\tau) = \mathbf{E}[a_{e_i}(\tau)a_{e_i}(\tau)] \tag{10}$$

$$= a_{e_i}(t_0)^2 e^{-2\beta_i(\tau - t_0)} + \frac{\tilde{Q}_i^{diag}}{2\beta_i} (1 - e^{-2\beta_i(\tau - t_0)})$$
(11)

Here, \tilde{Q}_i^{diag} is the *i*th element of the main diagonal of $\tilde{\mathbf{Q}}$. It can be seen from Eqn. (9) that each empirical acceleration a_{e_i} is exponentially correlated in time, and the inverse of β_i is the correlation time. The degree of autocorrelation is determined by the choice of \tilde{Q}_i^{diag} and β_i .⁶ For a finite value of \tilde{Q}_i^{diag} and $\beta_i = 0$, the model in Eqn. (8) reduces to a random walk process. On the other hand, the model approaches a zero-mean Gaussian white noise sequence as $\beta_i \to \infty$. Modeling the time correlation of the unmodeled accelerations allows DMC to achieve greater estimation accuracy than SNC when optimal values of $\tilde{\mathbf{Q}}$ and β are used.^{12,13,15} The direct estimate of the unmodeled accelerations may also be useful for improving the dynamical model in post-flight analyses.^{6,13} However, like SNC the required offline tuning makes it difficult to use DMC when the dynamical environment is poorly known a priori or the process noise is time-varying.

Covariance Matching Adaptive Filtering

Many CM techniques have been proposed to adaptively tune the measurement noise covariance, **R**, and the DT process noise covariance, $\mathbf{Q}^{2,29}$ Here, some commonly used CM techniques for estimating **Q** are derived. First, consider the linear time-varying system described by Eqn. (3) with DT measurements given by

$$\boldsymbol{z_k} = \mathbf{H_k} \boldsymbol{x_k} + \boldsymbol{\nu_k} \tag{12}$$

where \mathbf{H}_{k} is the measurement sensitivity matrix, and ν_{k} is the measurement noise. It is assumed that w_{k} and ν_{k} are zero-mean white Gaussian sequences with constant covariances $\mathbf{E}[w_{i}w_{j}^{T}] = \mathbf{Q}\delta_{ij}$ and $\mathbf{E}[\nu_{i}\nu_{j}^{T}] = \mathbf{R}\delta_{ij}$ respectively where δ_{ij} is the Kronecker delta function. The pre-fit measurement residual or measurement innovation at time step k is denoted by

$$\Delta_k^z = z_k - \mathbf{H}_k \hat{x}_{k|k-1} \tag{13}$$

$$=\mathbf{H}_{k}(\boldsymbol{x}_{k}-\hat{\boldsymbol{x}}_{k|k-1})+\boldsymbol{\nu}_{k}$$
(14)

where $\hat{x}_{k|k-1}$ is the mean state estimate at time step k taking into account all the measurements through time step k-1. The theoretical measurement innovation covariance is

$$\mathbf{S}_{k} = \mathbf{E}[\boldsymbol{\Delta}_{k}^{z} \boldsymbol{\Delta}_{k}^{z^{T}}]$$
(15)

$$=\mathbf{H}_{k}\mathbf{P}_{k|k-1}\mathbf{H}_{k}^{T}+\mathbf{R}_{k}$$
(16)

For an optimal Kalman filter, an unbiased empirical estimate of the measurement innovation covariance is given by $\hat{\mathbf{S}}_{k} = \Delta_{k}^{z} \Delta_{k}^{z^{T}}$.

The Kalman filter measurement update equation of the formal covariance can be written as

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_k \mathbf{S}_k \mathbf{K}_k^T \tag{17}$$

where $\mathbf{K}_{k} = \mathbf{P}_{k|k-1} \mathbf{H}_{k}^{T} \mathbf{S}_{k}^{-1}$ is the Kalman gain. Substituting Eqn. (6) into Eqn. (17) and solving for \mathbf{Q}_{k} yields

$$\mathbf{Q}_{k} = \mathbf{P}_{k|k} - \boldsymbol{\Phi}_{k} \mathbf{P}_{k-1|k-1} \boldsymbol{\Phi}_{k}^{T} + \mathbf{K}_{k} \mathbf{S}_{k} \mathbf{K}_{k}^{T}$$
(18)

Under the previous assumption that the DT process noise covariance is constant, \mathbf{Q}_k can be estimated using a sliding window of length N by setting the theoretical measurement innovation covariance equal to its empirical value which yields

$$\hat{\mathbf{Q}}_{k} = \frac{1}{N} \sum_{p=k-N}^{k-1} \left(\mathbf{P}_{p|p} - \boldsymbol{\Phi}_{p} \mathbf{P}_{p-1|p-1} \boldsymbol{\Phi}_{p}^{T} + \boldsymbol{\Delta}_{p}^{x} \boldsymbol{\Delta}_{p}^{x^{T}} \right)$$
(19)

Here,

$$\Delta_k^x = \mathbf{K}_k \Delta_k^z \tag{20}$$

is the state innovation at time step k. Eqn. (19) was originally derived by Myers and Tapley,^{29,33} with the difference being that here it is assumed that the DT process noise is known to be zero-mean. Interestingly, Fraser³² shows that maximum likelihood estimation (MLE) can also be used to obtain Eqn. (19) while Mohamed and Schwarz³¹ use MLE to derive a very similar result.

An obvious shortcoming of Eqn. (19) is that $\hat{\mathbf{Q}}_{k}$ may not be PSD. In order to be directly used in a Kalman filter, each $\hat{\mathbf{Q}}_{k}$ must be guaranteed PSD to be a valid covariance matrix and avoid losing the positive definiteness of the formal covariance in Eqn. (6). In practice, ad hoc methods are used to ensure that $\hat{\mathbf{Q}}_{k}$ is PSD. In the original derivation of Eqn. (20), Myers and Tapley^{29,33} suggest setting the diagonal of $\hat{\mathbf{Q}}_{k}$ equal to its absolute value. However, this results in a biased estimate and does not guarantee that $\hat{\mathbf{Q}}_{k}$ is PSD. The most common approach is to assume that the first two terms in Eqn. (19) are negligible at steady state. Under this assumption, Eqn. (19) reduces to^{31,32}

$$\hat{\mathbf{Q}}_{k} = \frac{1}{N} \sum_{p=k-N}^{k-1} \boldsymbol{\Delta}_{p}^{x} \boldsymbol{\Delta}_{p}^{x^{T}}$$
(21)

which ensures $\hat{\mathbf{Q}}_k$ is PSD. However, steady state conditions only guarantee that $\mathbf{P}_{p|p} = \mathbf{P}_{p-1|p-1}$, which is not a sufficient condition for $\mathbf{P}_{p|p} - \Phi_p \mathbf{P}_{p-1|p-1} \Phi_p^T = \mathbf{0}$. For a stable system, $\mathbf{P}_{p|p}$ is larger than $\Phi_p \mathbf{P}_{p-1|p-1} \Phi_p^T$ at steady state, and Eqn. (21) underestimates Q. System instability has the opposite effect and leads to an overestimate of Q. More rigorously, a sufficient condition for the first two terms in Eqn. (19) to add to zero is for the filter to be at steady state and the STM to be identity. Generally, the STM of an orbital element state remains close to identity for longer time steps than a Cartesian state because orbital element states vary more slowly in time. However, if the estimated state also contains force model parameters or the spacecraft is in a highly perturbed orbit, the STM may only be close to identity for very small time steps, even for orbital element states. Moreover, the measurement rate may be limited by measurement availability or computational resources. Another common assumption is that $\mathbf{K}_{k}(\frac{1}{N}\sum_{p=k-N}^{k-1}\Delta_{p}^{z}\Delta_{p}^{z^{T}})\mathbf{K}_{k}^{T}$ can replace $\frac{1}{N} \sum_{p=k-N}^{k-1} \Delta_p^x \Delta_p^{x^T}$ in Eqns. (19) and (21) if the filter is at steady state such that $\mathbf{K}_p = \mathbf{K}_k \ \forall \ k - N \le p < k$. Depending on the dimensionality of the state and measurement vectors either tors, it may be more computationally efficient to estimate the process noise covariance using either the measurement innovations, Δ^z , or state innovations, $\Delta^{x,7}$ Note that CM has a single tunable parameter, which is the length of the sliding window. This parameter should be chosen based on how quickly Q is expected to vary in time. A longer sliding window provides a more accurate estimate of Q when it is approximately constant. On the other hand, a shorter sliding window provides quicker adaptation of a time-varying \mathbf{Q}^{31}

There are several drawbacks to directly applying the aforementioned CM adaptive filtering techniques to orbit determination. First, the STM is often not close to identity and Eqn. (21) results in a biased estimate of the DT process noise covariance. Second, CM cannot accurately extrapolate the DT process noise covariance over a gap in measurements. As can be seen in Eqn. (5), a long measurement update interval due to a measurement outage will likely have a DT process noise covariance that is significantly different from previous shorter measurement update intervals given the same fidelity filter dynamics model. Depending on the measurement type and mission scenario, measurement outages may be long and frequent as is often the case in angles-only relative navigation.³⁶ Third, the aforementioned CM techniques do not explicitly take into account that process noise is due to CT unmodeled accelerations. Eqn. (21) can result in any PSD $\hat{\mathbf{Q}}_{\mathbf{k}}$, and there is no guarantee that a PSD $\widetilde{\mathbf{Q}}$ exists that relates to $\hat{\mathbf{Q}}_{\mathbf{k}}$ through Eqn. (5). Lastly, to the authors' knowledge there has been no work on CM adaptive filtering for systems with colored process noise. These shortcomings are overcome by two novel techniques presented in the following section.

ADAPTIVE AND DYNAMICALLY CONSTRAINED PROCESS NOISE ESTIMATION

This section describes how CM can be optimally combined with either SNC or DMC to overcome the limitations of the individual techniques. As illustrated in Figure 1, this results in two novel, adaptive, and dynamically constrained process noise algorithms called ASNC and ADMC. Unlike SNC and DMC, the proposed techniques are adaptive and well suited for scenarios where the process noise is time-varying and the dynamical environment is poorly known a priori. Furthermore, ASNC and ADMC are more accurate than CM because the first two terms in Eqn. (19) are not neglected and the estimated process noise covariance is constrained according to the underlying CT dynamical model. ADMC is able to achieve additional accuracy over CM and ASNC when the measurement rate and accuracy are sufficient to enable accurate tracking of the unmodeled accelerations. The proposed techniques are also able to accurately extrapolate the DT process noise covariance over measurement outages. The following two subsections describe how CM is optimally fused with SNC and DMC through a constrained weighted least squares minimization. In the next subsection, the weighting matrix used in the least squares optimization is derived. The final subsection describes how the weighted least squares problem can be efficiently solved.



Figure 1: Conceptual block diagram description of ASNC and ADMC.

Adaptive State Noise Compensation

Without loss of generality, the estimated state can be written as

$$\boldsymbol{x} = [\boldsymbol{x}_{\boldsymbol{s}}^T \ \boldsymbol{x}_{\boldsymbol{p}}^T]^T \tag{22}$$

where x_s is a vector comprising the estimated spacecraft state, and x_p is a vector containing all other estimated parameters. Considering the state ordering in Eqn. (22), the DT process noise covariance can be written in block matrix form as

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{ss} & \mathbf{Q}_{sp} \\ \mathbf{Q}_{sp}^T & \mathbf{Q}_{pp} \end{bmatrix}$$
(23)

where \mathbf{Q}_{ss} is the DT process noise covariance of the spacecraft state, \mathbf{Q}_{pp} is the DT process noise covariance of the other estimated parameters, and \mathbf{Q}_{sp} is the cross covariance. It is assumed that the CT process noise covariance $\widetilde{\mathbf{Q}}$ is constant over each measurement update interval. Consequently, Eqn. (5) becomes

$$\mathbf{Q}_{k} = \int_{t_{k-1}}^{t_{k}} \mathbf{\Phi}(t_{k}, \tau) \mathbf{\Gamma}(\tau) \widetilde{\mathbf{Q}}_{k} \mathbf{\Gamma}(\tau)^{T} \mathbf{\Phi}(t_{k}, \tau)^{T} d\tau$$
(24)

Due to the structure of Eqn. (24), \mathbf{Q}_k is guaranteed to be PSD if $\widetilde{\mathbf{Q}}_k$ is PSD.

At each time step, ASNC finds the PSD \mathbf{Q} that minimizes the difference in a weighted least squares sense between the elements of the spacecraft state DT process noise covariance \mathbf{Q}_{ss} obtained through Eqn. (24) and the corresponding CM estimate obtained from Eqn. (19). Since both the SNC modeled \mathbf{Q}_{ss} and the corresponding CM estimate are symmetric only the unique elements, which are contained in the lower triangular portions of the matrices, need to be fitted. The half-vectorization vech(·) indicates a vector composed of stacking the lower triangular elements of a matrix column-wise, which in this paper is also denoted by the superscript *vech*. For example,

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad \Rightarrow \quad \operatorname{vech}(\mathbf{M}) = \mathbf{M}^{vech} = \begin{bmatrix} M_{11} & M_{21} & M_{22} \end{bmatrix}^T \tag{25}$$

In Eqn. (24), the DT process noise covariance \mathbf{Q}_k is linear in \mathbf{Q}_k . To reduce computation time, $\mathbf{\widetilde{Q}}$ is assumed diagonal in this work. Under this assumption, $\mathbf{Q}_{ss_k}^{vech}$ can be expressed as

$$\mathbf{Q}_{ss_{k}}^{vech} = \mathbf{X}_{k} \widetilde{\mathbf{Q}}_{k}^{diag} \tag{26}$$

where the vector $\widetilde{\mathbf{Q}}_{k}^{diag}$ is the main diagonal of $\widetilde{\mathbf{Q}}_{k}$. The matrix \mathbf{X}_{k} is a function of Φ , Γ , and the length of the measurement update interval. To construct \mathbf{X}_{k} , it is necessary to compute the integral in Eqn. (24), which can always be done numerically. It is also straightforward to compute the integral analytically for linear time-invariant systems and some linear time-varying systems. At each time step, the adaptively tuned value of $\widetilde{\mathbf{Q}}^{diag}$ is found by solving the constrained weighted least squares minimization

$$\underset{\widetilde{\mathbf{Q}}^{diag}}{\arg\min} (\mathbf{X}_{k} \widetilde{\mathbf{Q}}^{diag} - \hat{\mathbf{Q}}_{ss_{k+1}}^{vech})^{T} \mathbf{W}_{k}^{-1} (\mathbf{X}_{k} \widetilde{\mathbf{Q}}^{diag} - \hat{\mathbf{Q}}_{ss_{k+1}}^{vech}) \quad \text{subject to} \quad \widetilde{\mathbf{Q}}^{diag} \ge \mathbf{0}$$
(27)

Here \mathbf{W}_k is the theoretical covariance of $\hat{\mathbf{Q}}_{ss_{k+1}}^{vech}$, which weights the solution of Eqn. (27) more heavily towards components of $\hat{\mathbf{Q}}_{ss_{k+1}}^{vech}$ that are known with more certainty. The inequality constraint in Eqn. (27) ensures $\widetilde{\mathbf{Q}}$ is PSD. The matrix $\widetilde{\mathbf{Q}}_{k+1}$ is set equal to the diagonal $\widetilde{\mathbf{Q}}$ that solves

the optimization in Eqn. (27). Then \mathbf{Q}_{k+1} is computed through Eqn. (24), which accurately adapts the DT process noise covariance according to the length of the measurement update interval. This appropriately enlarges the DT process noise covariance when the measurement update interval is large compared to previous update intervals due to a measurement outage. Any measurement update interval corresponding to a measurement outage should be excluded from the sliding window average shown in Eqn. (19) to avoid biasing the estimate of $\tilde{\mathbf{Q}}$. Note that the time index of $\hat{\mathbf{Q}}_{ss}^{vech}$ in Eqn. (27) is k+1 in order to estimate $\tilde{\mathbf{Q}}$ using the most recent available data. When constructing $\hat{\mathbf{Q}}_{ssk+1}^{vech}$ from Eqn. (19), the sliding window average includes data at time step k, which is available since the estimated $\tilde{\mathbf{Q}}$ is not used to compute \mathbf{Q} until time step k + 1. Like CM, the length of the sliding window is the only tunable parameter in ASNC. This parameter should be chosen based on how quickly the process noise covariance is expected to vary in time.

Adaptive Dynamic Model Compensation

In ADMC, the state is augmented with empirical accelerations. Without loss of generality, the state and corresponding DT process noise covariance can be written as

$$\boldsymbol{x} = [\boldsymbol{x}_{\boldsymbol{s}}^{T} \ \boldsymbol{a}_{\boldsymbol{e}}^{T} \ \boldsymbol{x}_{\boldsymbol{p}}^{T}]^{T}, \quad \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{ss} & \mathbf{Q}_{se} & \mathbf{Q}_{sp} \\ \mathbf{Q}_{se}^{T} & \mathbf{Q}_{ee} & \mathbf{Q}_{ep} \\ \mathbf{Q}_{sp}^{T} & \mathbf{Q}_{ep}^{T} & \mathbf{Q}_{pp} \end{bmatrix}$$
(28)

where \mathbf{Q}_{ee} is the DT process noise covariance of the empirical accelerations. In ADMC, $\widetilde{\mathbf{Q}}$ is estimated at each time step by solving the minimization in Eqn. (27). In this equation, \mathbf{X}_k is different when using ADMC or ASNC since Φ and Γ are altered when the state is augmented with empirical accelerations. Although \mathbf{Q}_{ss} , \mathbf{Q}_{se} , and \mathbf{Q}_{ee} are all linear functions of $\widetilde{\mathbf{Q}}$ in Eqn. (24), only the CM estimate of \mathbf{Q}_{ss} is used in Eqn. (27) to estimate $\widetilde{\mathbf{Q}}$ because simulations show that it is detrimental to include CM estimates of \mathbf{Q}_{ee} and \mathbf{Q}_{se} . This is probably due to the significant difference between the true dynamics of the unmodeled accelerations and the modeled empirical acceleration dynamics shown in Eqn. (7).

For ADMC, solving Eqn. (27) at each time step leads to oscillations in \mathbf{Q} . These oscillations likely occur because Eqn. (27) does not take into account how a change in \mathbf{Q} affects the formal covariance of the empirical accelerations which in turn influences the time updated formal covariance of the spacecraft state at the following time step. This creates a one time step delay in the effective formal uncertainty added to the spacecraft state due to the empirical accelerations. The oscillations in the estimate history of \mathbf{Q} can be smoothed by utilizing a forgetting factor, α , which has similarly been used for CM.³² The filter can then be implemented as

$$\widetilde{\mathbf{Q}}_{k} = (1 - \alpha)\widetilde{\mathbf{Q}}_{k-1} + \alpha\widetilde{\mathbf{Q}}_{k}^{*}$$
(29)

where $\widetilde{\mathbf{Q}}_{k}^{*}$ is the diagonal matrix that minimizes Eqn. (27), and $0 < \alpha \leq 1$ is a selected constant. The value of α is a tunable parameter that provides greater smoothing for smaller values. The range $0.01 \leq \alpha \leq 0.05$ has been effective for the case studies presented in this paper and can be used as a guideline. After calculating $\widetilde{\mathbf{Q}}_{k}$ through Eqn. (29), Eqn. (24) is used to compute \mathbf{Q}_{k+1} under the assumption that $\widetilde{\mathbf{Q}}_{k+1} = \widetilde{\mathbf{Q}}_{k}$. Like ASNC, this accurately adapts the DT process noise covariance according to the length of the measurement update interval.

Weighting Matrix Derivation

This section derives the least squares weighting matrix \mathbf{W}_k , which is the theoretical covariance matrix of $\hat{\mathbf{Q}}_{ss_{k+1}}^{vech}$. The sample variance of each element of $\hat{\mathbf{Q}}_{k+1}$ as shown in Eqn. (19) is simply equal to the sample variance of each element of $\frac{1}{N}\sum_{p=k-N+1}^{k} \Delta_p^x \Delta_p^{x^T}$ since the other terms in the equation are known constants. Both Kailath³⁷ and Mehra² provide proofs that the measurement innovations, Δ^z , are a zero-mean white Gaussian process for an optimal Kalman filter at steady state. Therefore, it is apparent from Eqn. (20) that each Δ_k^x is normally distributed because it is a linear combination of normally distributed random variables. Applying the expectation operator to Eqn. (20) also reveals that each Δ_k^x is zero-mean. Furthermore, assuming that the filter is at steady state such that the Kalman gain is constant, it is easily shown that Δ^x is uncorrelated in time by

$$\mathbf{E}[\boldsymbol{\Delta}_{t}^{\boldsymbol{x}} \boldsymbol{\Delta}_{\tau}^{\boldsymbol{x}^{T}}] = \mathbf{K} \mathbf{E}[\boldsymbol{\Delta}_{t}^{\boldsymbol{z}} \boldsymbol{\Delta}_{\tau}^{\boldsymbol{z}^{T}}] \mathbf{K}^{T}$$
(30)

$$= \mathbf{0} \ \forall \ t \neq \tau \tag{31}$$

since $E[\Delta_t^z \Delta_\tau^{z^T}] = 0 \quad \forall t \neq \tau.^{2,37}$ Note, Myers and Tapley state the assumption that Δ^x is not correlated in time in the original derivation of Eqn. (19).^{29,33} The theoretical covariance of Δ_k^x is

$$\Sigma_{k} = \mathrm{E}[\Delta_{k}^{x} \Delta_{k}^{x^{T}}]$$
(32)

$$=\mathbf{K}_{k}\mathbf{S}_{k}\mathbf{K}_{k}^{T}$$
(33)

Now two identities are employed to derive \mathbf{W} . First, the covariance of two sums of random variables is given by

$$\operatorname{Cov}\left(\sum_{i=1}^{n_i} X_i, \sum_{j=1}^{n_j} Y_j\right) = \sum_i^{n_i} \sum_j^{n_j} \operatorname{Cov}(X_i, Y_j)$$
(34)

where X_i and Y_j are random variables. Second, Isserlis' theorem states that for normally distributed random variables $X_1, ..., X_4$,

$$E[X_1X_2X_3X_4] = E[X_1X_2]E[X_3X_4] + E[X_1X_3]E[X_2X_4] + E[X_1X_4]E[X_2X_3]$$
(35)

Using these two identities, the covariance matrix of $\hat{\mathbf{Q}}_{ss_{k+1}}^{vech}$ can be constructed from

$$\operatorname{Cov}(\hat{\mathbf{Q}}_{k+1_{ij}}, \hat{\mathbf{Q}}_{k+1_{mn}}) = \operatorname{Cov}\left(\frac{1}{N} \sum_{p=k-N+1}^{k} \Delta_{p_i}^x \Delta_{p_j}^x, \frac{1}{N} \sum_{p=k-N+1}^{k} \Delta_{p_m}^x \Delta_{p_n}^x\right)$$
(36)

$$=\frac{1}{N^{2}}\operatorname{Cov}\left(\sum_{p=k-N+1}^{k} \Delta_{p_{i}}^{x} \Delta_{p_{j}}^{x}, \sum_{p=k-N+1}^{k} \Delta_{p_{m}}^{x} \Delta_{p_{n}}^{x}\right)$$
(37)

$$= \frac{1}{N^2} \operatorname{Cov} \sum_{p=k-N+1}^{\kappa} \operatorname{Cov}(\Delta_{p_i}^x \Delta_{p_j}^x, \Delta_{p_m}^x \Delta_{p_n}^x)$$
(38)

$$=\frac{1}{N^2}\sum_{p=k-N+1}^{k} \left(\mathbf{E}[\boldsymbol{\Delta}_{p_i}^{x}\boldsymbol{\Delta}_{p_j}^{x}\boldsymbol{\Delta}_{p_m}^{x}\boldsymbol{\Delta}_{p_n}^{x}] - \mathbf{E}[\boldsymbol{\Delta}_{p_i}^{x}\boldsymbol{\Delta}_{p_j}^{x}]\mathbf{E}[\boldsymbol{\Delta}_{p_m}^{x}\boldsymbol{\Delta}_{p_n}^{x}] \right)$$
(39)

$$=\frac{1}{N^2}\sum_{p=k-N+1}^{k} (\Sigma_{p_{im}}\Sigma_{p_{jn}} + \Sigma_{p_{in}}\Sigma_{p_{jm}})$$
(40)

Applying Eqn. (34) to Eqn. (37) and recalling the assumption that Δ^x is uncorrelated in time yields Eqn. (38). The identity in Eqn. (35) can be applied to Eqn. (39) to yield Eqn. (40).

If W is approximated as diagonal, it can be expressed as

$$\mathbf{W}_{k} = \frac{1}{N^{2}} \sum_{p=k-N+1}^{k} \mathcal{W}_{p}$$
(41)

where

$$\mathcal{W} = \operatorname{diag}(\operatorname{vech}(\overline{\Sigma})) \tag{42}$$

$$\overline{\Sigma} = \Sigma_{ss}^{\circ 2} + \Sigma_{ss}^{diag} \Sigma_{ss}^{diag^T}$$
(43)

Here, diag(·) denotes a square diagonal matrix whose main diagonal is the vector inside the parenthesis. The matrix Σ_{ss} is the theoretical covariance of the portion of the state innovation corresponding to the spacecraft state, which is a submatrix of Σ . For a six dimensional spacecraft state and the state orderings shown in Eqns. (22) and (28), $\Sigma_{ss} \in \mathbb{R}^{6\times 6}$ is the first six rows and columns of Σ . The vector Σ_{ss}^{diag} is the main diagonal of Σ_{ss} . The Hadamard power, °, denotes an elementwise power. For example, $\mathbf{A} = \mathbf{B}^{\circ 2}$ indicates that $A_{ij} = B_{ij}^2$. If the filter is at steady state such that $\Sigma_p = \Sigma_k \forall k - N , Eqn. (41) reduces to$

$$\mathbf{W}_{k} = \frac{1}{N} \boldsymbol{\mathcal{W}}_{k} \tag{44}$$

Note that the factor of $\frac{1}{N^2}$ in Eqns. (40) and (41) and the factor of $\frac{1}{N}$ in Eqn. (44) can be dropped since a constant factor does not change the solution of the least squares minimization in Eqn. (27).

Constrained Weighted Least Squares Solution

The ASNC and ADMC algorithms are summarized in Table 1. On line four, $\tilde{\mathbf{Q}}$ is estimated through a constrained weighted least squares optimization, which approximates the maximum likelihood estimate. Given a linear system as shown in Eqns. (3) and (12) and assuming an optimal Kalman filter with constant \mathbf{Q} , Eqn. (19) provides an unbiased estimate of the DT process noise covariance.^{29,33} According to the Central Limit Theorem, the probability distribution of each element of the last term in Eqn. (19) approaches a normal distribution as the length of the sliding window increases. Therefore, the weighted least squares solution obtained through Eqn. (27) approaches the maximum likelihood estimate of $\tilde{\mathbf{Q}}$ as the length of the sliding window increases.

Table 1: Summary of the ASNC and ADMC algorithms.

- 1: Use CM to compute $\hat{\mathbf{Q}}_{ss_{k+1}}$ using Eqn. (19)
- 2: Calculate Σ_k through Eqn. (33)
- 3: Compute \mathbf{W}_{k} using Eqns. (41-43)
- 4: Determine $\widetilde{\mathbf{Q}}_{k}^{*}$ by solving the optimization in Eqn. (27)
- 5: For ADMC calculate $\widetilde{\mathbf{Q}}_{k}$ through Eqn. (29), and for ASNC $\widetilde{\mathbf{Q}}_{k} = \widetilde{\mathbf{Q}}_{k}^{*}$
- 6: Set $\widetilde{\mathbf{Q}}_{k+1} = \widetilde{\mathbf{Q}}_k$
- 7: Compute Q_{k+1} using Eqn. (24)

The optimization in Eqn. (27) is a quadratic program with a non-negativity constraint on the optimization variables. In general, this optimization can be efficiently solved using active set or interior point methods,^{38–41} and applicable solvers are readily available in various programming languages. However, Eqn. (27) can be solved with a significant reduction in computation time when **W** is approximated as diagonal, the spacecraft state is represented by Cartesian coordinates, and the CT process noise ϵ is expressed in the same frame as the spacecraft state. Under these conditions, each element of $\widetilde{\mathbf{Q}}^{diag}$ can be solved for independently without any matrix inversions or decompositions. This computationally efficient solution is described in the remainder of this subsection for both ASNC and ADMC.

Adaptive State Noise Compensation. Consider the state shown in Eqn. (22) where the spacecraft state is represented by Cartesian coordinates as

$$\boldsymbol{x}_{\boldsymbol{s}} = [\boldsymbol{r}^T \ \boldsymbol{v}^T]^T \tag{45}$$

Here r and v are the spacecraft position and velocity vectors respectively expressed in an inertial frame. Recall the corresponding DT process noise covariance in block matrix form shown in Eqn. (23). To avoid the computational cost of numerically evaluating Eqn. (24), an analytical solution derived by Myers³³ can be utilized which is given by

$$\mathbf{Q}_{ss_{k}} = \begin{bmatrix} \frac{1}{3}\Delta t^{3}\widetilde{\mathbf{Q}}_{k} & \frac{1}{2}\Delta t^{2}\widetilde{\mathbf{Q}}_{k} \\ \frac{1}{2}\Delta t^{2}\widetilde{\mathbf{Q}}_{k} & \Delta t\widetilde{\mathbf{Q}}_{k} \end{bmatrix}$$
(46)

when the CT process noise ϵ is expressed in the same frame as the spacecraft state. Here $\Delta t = t_k - t_{k-1}$ is the length of the measurement update interval. Although never explicitly stated by Myers^{13,29,33}, Eqn. (46) is only an approximate solution to Eqn. (24). This approximation assumes the process noise does not perturb the state enough to change the modeled accelerations over the measurement update interval. Eqn. (46) more closely matches the true solution of Eqn. (24) as Δt decreases. Notice that each element of \mathbf{Q}_{ss} is a function of a single element of \mathbf{Q} in Eqn. (46). Assuming \mathbf{W} is diagonal, the optimization in Eqn. (27) can be written as

$$\underset{\widetilde{\mathbf{Q}}^{diag}}{\text{minimize}} \sum_{i=1}^{3} (\overline{\mathbf{X}}(i)\widetilde{Q}_{i}^{diag} - \mathbf{b}(i))^{T} \overline{\mathbf{W}}(i)^{-1} (\overline{\mathbf{X}}(i)\widetilde{Q}_{i}^{diag} - \mathbf{b}(i)) \quad \text{subject to} \quad \widetilde{\mathbf{Q}}^{diag} \ge \mathbf{0} \quad (47)$$

where \widetilde{Q}_{i}^{diag} is the *i*th element of $\widetilde{\mathbf{Q}}^{diag}$ and

$$\overline{\boldsymbol{X}}(i) = \begin{bmatrix} \frac{1}{3}\Delta t^3 \\ \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix}, \quad \boldsymbol{b}(i) = \begin{bmatrix} \hat{Q}_{k+1_{i,i}} \\ \hat{Q}_{k+1_{3+i,i}} \\ \hat{Q}_{k+1_{3+i,3+i}} \end{bmatrix}, \quad \overline{\mathbf{W}}(i) = \operatorname{diag}\left(\sum_{p=k-N+1}^k \begin{bmatrix} \overline{\Sigma}_{p_{i,i}} \\ \overline{\Sigma}_{p_{3+i,i}} \\ \overline{\Sigma}_{p_{3+i,3+i}} \end{bmatrix}\right) \quad (48)$$

The objective function in Eqn. (47) is the sum of three independent quadratic functions which are each a function of a single optimization variable. Consequently, each element of $\tilde{\mathbf{Q}}^{diag}$ can be solved for independently by finding the unconstrained solution and setting any negative elements equal to zero. The solution to the optimization in Eqn. (47) can therefore be written as

$$\widetilde{Q}_{i}^{diag} = \max\left(0, \ \frac{\overline{\mathbf{X}}(i)^{T}\overline{\mathbf{W}}(i)^{-1}\boldsymbol{b}(i)}{\overline{\mathbf{X}}(i)^{T}\overline{\mathbf{W}}(i)^{-1}\overline{\mathbf{X}}(i)}\right)$$
(49)

where $\overline{\mathbf{W}}(i)^{-1}$ can be computed as an element-wise inverse because it is a diagonal matrix. Note that Eqn. (49) is an exact solution of the optimization shown in Eqn. (47) and is computationally efficient because it does not require any matrix inversions or matrix decompositions. The estimated value of $\widetilde{\mathbf{Q}}$ computed through Eqn. (49) is used to calculate $\mathbf{Q}_{ss_{k+1}}$ through Eqn. (46).

Adaptive Dynamic Model Compensation. Consider the state and corresponding block matrix DT process noise covariance as shown in Eqn. (28) where the spacecraft state is represented with Cartesian coordinates expressed in an inertial frame as shown in Eqn. (45). In this case it is straightforward to compute an exact analytical solution of the integral in Eqn. (24). This was first demonstrated by Cruickshank¹⁶ who showed the spacecraft state DT process noise covariance given by Eqn. (24) can be written as^{6, 16, 17}

$$\mathbf{Q}_{ss_{k}} = \begin{bmatrix}
\mathbf{C}_{1,1} \circ \widetilde{\mathbf{Q}}_{k} & \mathbf{C}_{2,1} \circ \widetilde{\mathbf{Q}}_{k} & \mathbf{C}_{3,1} \circ \widetilde{\mathbf{Q}}_{k} \\
\mathbf{C}_{2,1} \circ \widetilde{\mathbf{Q}}_{k} & \mathbf{C}_{2,2} \circ \widetilde{\mathbf{Q}}_{k} & \mathbf{C}_{3,2} \circ \widetilde{\mathbf{Q}}_{k} \\
\mathbf{C}_{3,1} \circ \widetilde{\mathbf{Q}}_{k} & \mathbf{C}_{3,2} \circ \widetilde{\mathbf{Q}}_{k} & \mathbf{C}_{3,3} \circ \widetilde{\mathbf{Q}}_{k}
\end{bmatrix}$$

$$C_{1,1_{i}} = \frac{\Delta t^{3}}{3\beta_{i}^{2}} - \frac{\Delta t^{2}}{\beta_{i}^{3}} + \frac{\Delta t}{\beta_{i}^{4}} (1 - 2e^{-\beta_{i}\Delta t}) + \frac{1}{2\beta_{i}^{5}} (1 - e^{-2\beta_{i}\Delta t})$$

$$C_{2,1_{i}} = \frac{\Delta t^{2}}{2\beta_{i}^{2}} - \frac{\Delta t}{\beta_{i}^{3}} (1 - e^{-\beta_{i}\Delta t}) + \frac{1}{\beta_{i}^{4}} (1 - e^{-\beta_{i}\Delta t}) - \frac{1}{2\beta_{i}^{4}} (1 - e^{-2\beta_{i}\Delta t})$$

$$C_{3,1_{i}} = \frac{1}{2\beta_{i}^{3}} (1 - e^{-2\beta_{i}\Delta t}) - \frac{\Delta t}{\beta_{i}^{2}} e^{-\beta_{i}\Delta t}$$

$$C_{2,2_{i}} = \frac{\Delta t}{\beta_{i}^{2}} - \frac{2}{\beta_{i}^{3}} (1 - e^{-\beta_{i}\Delta t}) + \frac{1}{2\beta_{i}^{3}} (1 - e^{-2\beta_{i}\Delta t})$$

$$C_{3,2_{i}} = \frac{1}{2\beta_{i}^{2}} (1 + e^{-2\beta_{i}\Delta t}) - \frac{1}{\beta_{i}^{2}} e^{-\beta_{i}\Delta t}$$

$$C_{3,3_{i}} = \frac{1}{2\beta_{i}} (1 - e^{-2\beta_{i}\Delta t})$$

when β and $\widetilde{\mathbf{Q}}$ are diagonal matrices, and the CT process noise ϵ is expressed in the same frame as the spacecraft state. Here $\mathbf{C}_{1,1}, \ldots, \mathbf{C}_{3,3} \in \mathbb{R}^{3\times3}$ are each a diagonal matrix, and $C_{1,1_i}$ and β_i refer to the *i*th diagonal element of the matrices $\mathbf{C}_{1,1}$ and β respectively. The Hadamard product \circ is used in Eqn. (50) to denote element-wise multiplication. For example, $\mathbf{A} = \mathbf{B} \circ \mathbf{C}$ indicates that $A_{ij} = B_{ij}C_{ij}$. Similar to ASNC, each element of \mathbf{Q}_{ss} is a function of only a single element of $\widetilde{\mathbf{Q}}$. Approximating \mathbf{W} as diagonal, the optimization in Eqn. (27) can be rewritten as Eqn. (47) where $\mathbf{b}(i)$ and $\overline{\mathbf{W}}(i)$ are defined in Eqn. (48) and

$$\overline{\mathbf{X}}(i) = \begin{bmatrix} C_{1,1_i} & C_{2,1_i} & C_{2,2_i} \end{bmatrix}^T$$
(51)

The solution of $\widetilde{\mathbf{Q}}$ given by Eqn. (49) is used to compute $\mathbf{Q}_{ss_{k+1}}$ through Eqn. (50).

CASE STUDY I - PARTICLE IN ONE DIMENSION

This section demonstrates the benefits of the developed algorithms over SNC, DMC, and CM through a linear system since linearity is assumed in the development of each of these techniques. Note that a similar example is used by several authors to delineate SNC and DMC.^{6,16,17} Consider a particle moving along the x-axis with an initial position x_0 and velocity \dot{x}_0 . The particle is subject to an unknown perturbing acceleration in the x-direction given by

$$a_p = \cos\left(\frac{\pi}{5}t\right) \,\,\mathrm{m/s^2} \tag{52}$$

A Kalman filter is used to estimate the particle position and velocity over 240 s. Range and rangerate measurements are taken from the origin to the particle every 0.1 s and are corrupted by zeromean white Gaussian noise with standard deviations of 2 m and 0.1 m/s respectively. Note that for this one dimensional system, $\tilde{\mathbf{Q}}$ is a scalar. CM as described by Eqn. (21) as well as ASNC and ADMC are each utilized with a sliding window of 30 time steps, which is a 3 s interval. Adaptation of the process noise covariance does not commence until the 31^{st} filter call when the entire sliding window is filled. The performance of each considered process noise technique is characterized through Monte Carlo simulations where the initial $1-\sigma$ formal uncertainties provided to the filter in position and velocity are 1.8 m and 150 mm/s. In each simulation, the measurements are independently corrupted, and the initial error in position and velocity is randomly sampled according to the initial formal covariance provided to the filter. For DMC and ADMC, the empirical acceleration is initialized as zero. To reduce the memory required by ASNC and ADMC, Eqn. (44) is used to approximate the weighting matrix. This simplification provides nearly identical performance to the case when the weighting matrix is computed through Eqn. (41).

Adaptive State Noise Compensation

The estimated state is given by

$$\boldsymbol{x} = [x \ \dot{x}]^T \tag{53}$$

The plant matrix, process noise mapping matrix, STM, and measurement sensitivity matrix are

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{\Gamma} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{\Phi}(t, t_0) = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
(54)

Since this is a linear time invariant system, it is straightforward to analytically evaluate the integral in Eqn. (24) to determine an exact expression for the DT process noise covariance, which is given by^{6}

$$\mathbf{Q}_{k} = \widetilde{Q}_{k} \begin{bmatrix} \frac{1}{3} \Delta t^{3} & \frac{1}{2} \Delta t^{2} \\ \frac{1}{2} \Delta t^{2} & \Delta t \end{bmatrix}$$
(55)

ASNC adaptively tunes \widetilde{Q} at each time step by solving the weighted least squares optimization in Eqn. (27). The estimated value of \widetilde{Q} is then used to compute \mathbf{Q}_{k+1} through Eqn. (55). This equation illustrates how ASNC constrains the DT process noise covariance according to the underlying CT dynamical model, which assumes \widetilde{Q} is constant over the measurement update interval. The constraint on the DT process noise covariance can be seen more clearly by analyzing its eigenvalue decomposition. The eigenvectors and the ratio of eigenvalues of \mathbf{Q} are fixed. The chosen value of \widetilde{Q} only scales the magnitudes of the eigenvalues of \mathbf{Q} . Therefore, the shape and orientation of the 1- σ uncertainty ellipse associated with \mathbf{Q} are fixed, and the chosen value of \widetilde{Q} only scales that ellipse.

Adaptive Dynamic Model Compensation

For ADMC, the estimated state is augmented with a single empirical acceleration in the xdirection. The new state vector is given by

$$\boldsymbol{x} = [x \ \dot{x} \ a_e]^T \tag{56}$$

The plant matrix and process noise mapping matrix are

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\beta \end{bmatrix}, \quad \mathbf{\Gamma} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
(57)

The parameters β and α are set to 0.005 s⁻¹ and 0.02 respectively for this case study. The STM and measurement sensitivity matrix are

$$\mathbf{\Phi}(t,t_0) = \begin{bmatrix} 1 & \Delta t & \frac{1}{\beta}\Delta t + \frac{1}{\beta^2}(e^{-\beta\Delta t} - 1) \\ 0 & 1 & \frac{1}{\beta}(1 - e^{-\beta\Delta t}) \\ 0 & 0 & e^{-\beta\Delta t} \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
(58)

Evaluating Eqn. (24) yields

$$\mathbf{Q}_{k} = \tilde{Q}_{k} \begin{bmatrix} C_{1,1} & C_{2,1} & C_{3,1} \\ C_{2,1} & C_{2,2} & C_{3,2} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix}$$
(59)

Note that $C_{1,1}, \ldots, C_{3,3}$ are scalars for this one dimensional system and are defined in Eqn. (50). ADMC adaptively tunes \tilde{Q} at each time step by solving Eqn. (27). The solution to Eqn. (27), \tilde{Q}_{k}^{*} , is used to compute \tilde{Q}_{k+1} through Eqn. (29), which is then used to calculate \mathbf{Q}_{k+1} through Eqn. (59). As was the case with ASNC, the eigenvectors and ratio of eigenvalues of \mathbf{Q} are fixed due to the underlying CT dynamical model, which assumes \tilde{Q} is constant over the measurement update interval. The shape and orientation of the 1- σ uncertainty ellipsoid associated with \mathbf{Q} as shown in Eqn. (59) are fixed, and the chosen value of \tilde{Q} only scales that ellipsoid.

Results

Figure 2 shows the estimation mean absolute error (MAE) when using SNC, DMC, ASNC, and ADMC as a function of the initial guess of Q. The MAE when process noise is not modeled and when using CM is also shown in Figure 2 for reference. Each MAE is computed over the last 45 s of simulation and averaged over 1000 Monte Carlo simulations. As expected, not modeling process noise leads to filter inconsistency and large estimation error. SNC and DMC perform well when an optimal value of \widetilde{Q} is used. However, when the value of \widetilde{Q} is suboptimal the estimation errors become large, and the filter can become inconsistent. Interestingly, DMC is more robust than SNC to a poor choice of Q^{6} CM maintains filter consistency, but has large position errors. Remarkably, ASNC and ADMC maintain filter consistency and achieve 80% lower position error than CM regardless of the initial value of Q. ADMC provides a 20% reduction in velocity error over ASNC because the measurement rate is sufficiently high and the measurements are accurate enough for the filter to track the unmodeled acceleration well. Acceleration tracking and filter convergence plots are shown in Figures 4 and 5 in the appendix. The average run time per filter call when using a fixed process noise covariance was 46 µs for a MATLAB implementation on a 4 GHz Intel Core i7-6700 processor. Utilizing CM, ASNC, and ADMC respectively incurred an additional 25%, 74%, and 79% increase in computation time.

CASE STUDY II - FORMATION FLYING ABOUT ASTEROID

This section further validates the developed process noise techniques and demonstrates their performance by applying them to the autonomous navigation of two spacecraft orbiting the asteroid 433 Eros. Eros is utilized as the target asteroid because accurate shape and gravity models are available from the NEAR mission which ended in 2001.^{42–44} More recent asteroid missions such as OSIRIS-REx⁴⁵ and Hayabusa2⁴⁶ demonstrate the space community's continued interest in asteroids. There is a focus on asteroids for several reasons including science, mining, and planetary defense.^{47–50} Due to light time delay and limited ground-based resources such as NASA's Deep Space Network,



Figure 2: MAE in position, velocity, and acceleration for different process noise techniques as a function of the initial value of \tilde{Q}_0 . The units of \tilde{Q}_0 are m/s^{3/2} for SNC and m/s^{5/2} for DMC.⁶

there have been efforts to increase spacecraft autonomy for asteroid missions.^{4,51–54} The proposed process noise techniques enhance spacecraft autonomy and are well suited for the challenges of asteroid missions such as limited a priori knowledge of the dynamical environment and time-varying process noise.

An unscented Kalman filter (UKF)⁵⁵ is used to estimate the Cartesian states of the chief and deputy spacecraft through interspacecraft radio-frequency range and range-rate measurements as well as spacecraft camera pixel measurements of optical navigation (OpNav)^{52,53,56} features on the asteroid surface. This is consistent with the recently proposed ANS mission architecture.^{3,4} The performance of ASNC and ADMC as well as the CM technique given by Eqn. (21) are each characterized through 1000 Monte Carlo simulations. In each simulation, the measurements are independently corrupted by zero-mean Gaussian white noise, and the initial estimate error in position and velocity is randomly sampled according to the initial formal covariance provided to the filter. The empirical accelerations are initialized as zero for ADMC. Each simulation lasts for four orbit periods where the orbit period is approximately 20.9 hours.

Reference Truth

The reference truth spacecraft orbits are generated through high-fidelity numerical integration including the effects of the gravity field of Eros up to degree and order 15, third body effects from the sun, and solar radiation pressure (SRP). In modeling SRP, the area to mass ratio and reflectivity coefficient of each spacecraft are constant. The initial osculating Keplerian orbital elements of the chief are $[a_c \ e_c \ i_c \ \Omega_c \ \omega_c \ M_c] = [40 \text{ km } 0.01 \ 95^\circ \ 0^\circ \ 0^\circ]$. These orbital elements are defined with respect to an inertial frame centered at the asteroid center of mass. The z-axis of this frame is aligned with the mean spin axis of the asteroid, the x-axis is aligned with the asteroid prime meridian at the epoch J2000, and the y-axis completes the right-handed triad. In order to achieve passive collision avoidance between spacecraft, E/I vector separation is used to select the initial quasi-nonsingular relative orbital elements (ROE) of the deputy.⁸ The initial osculating ROE of the deputy multiplied by the chief semi-major axis are $a_c[\delta a \ \delta \lambda \ \delta e_x \ \delta e_y \ \delta i_x \ \delta i_y] = [0 \ 5 \ 0 \ 2 \ 0 \ 2]$ km. The ROE are defined

in terms of the chief and deputy spacecraft Keplerian orbital elements as^{7,8,57}

$$\begin{bmatrix} \delta a \\ \delta \lambda \\ \delta e_x \\ \delta e_y \\ \delta i_x \\ \delta i_y \end{bmatrix} = \begin{bmatrix} (a_d - a_c)/a_c \\ u_d - u_c + (\Omega_d - \Omega_c)\cos(i_c) \\ e_d\cos(w_d) - e_c\cos(w_c) \\ e_d\sin(w_d) - e_c\sin(w_c) \\ i_d - i_c \\ (\Omega_d - \Omega_c)\sin(i_c) \end{bmatrix}$$
(60)

Here u = M + w is the mean argument of latitude, and the subscripts c and d indicate the chief and deputy respectively.

Idealized interspacecraft range and range-rate measurements are given by

$$\rho = ||\boldsymbol{\rho}|| \qquad \dot{\rho} = \frac{\dot{\boldsymbol{\rho}} \cdot \boldsymbol{\rho}}{||\boldsymbol{\rho}||} \tag{61}$$

where $\rho = r_d - r_c$ is the position vector of the deputy with respect to the chief. Pixel measurements, u and v, taken by cameras onboard the chief and deputy spacecraft of visible OpNav features on the asteroid surface are given by the pin hole camera model

$$\begin{bmatrix} uw \\ vw \\ w \end{bmatrix} = \hat{\mathbf{K}}_{ACI \to CF} \begin{bmatrix} \mathbf{R} \\ ACAF \to ACI & -\mathbf{r} \end{bmatrix} \begin{bmatrix} \mathbf{L} \\ 1 \end{bmatrix}$$
(62)

Here *r* is the spacecraft position vector expressed in the Asteroid Centered Inertial (ACI) frame, which is aligned with the J2000 reference frame. The OpNav landmark position expressed in the Asteroid Centered Asteroid Fixed (ACAF) frame is denoted by *L*. Vectors expressed in the ACAF frame are expressed in the ACI frame through multiplication with the rotation matrix $\underset{ACAF \to ACI}{\mathbf{R}}$, and vectors expressed in the ACI frame are expressed in the Camera Fixed (CF) frame through the rotation matrix $\underset{ACIF \to CF}{\mathbf{R}}$. The matrix of known camera intrinsic parameters $\hat{\mathbf{K}}$ is defined by

$$\hat{\mathbf{K}} = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}$$
(63)

where f_x and f_y indicate the camera focal length divided by the pixel pitch in the CF frame x and y directions respectively. The vector $\mathbf{c} = [c_x c_y]^T$ denotes the principal point in units of pixels. The simulated camera properties are consistent with the OSIRIS-REx NavCam⁵⁸ as shown in Table 2. A set of 100 points on the asteroid surface were selected as OpNav features so that multiple OpNav features are visible at each time step. Pixel measurements of an OpNav feature taken from a spacecraft camera are only available if the feature is directly illuminated by the sun, is within the camera field of view, and is not blocked from the camera view by the asteroid. The range, rangerate, and pixel measurements are simulated every five minutes and are each corrupted by zero-mean Gaussian white noise with standard deviations of 10 cm, 1 mm/s, and 0.5 pixels respectively.

	Table 2: Simulated camera properties.				
mera	Number of Pixels	Field of View	F		

Camera	Number of Pixels	Field of View	Focal Length
OSIRIS-REx NavCam ⁵⁸	2592x1944	44°x32°	7.6 mm

Filter

The UKF estimated state is

$$\boldsymbol{x} = [\boldsymbol{r_c}^T \ \boldsymbol{v_c}^T \ \boldsymbol{r_d}^T \ \boldsymbol{v_d}^T]^T$$
(64)

where r and v indicate spacecraft position and velocity vectors respectively expressed in the ACI frame. The subscripts c and d respectively denote the chief and deputy spacecraft. The filter dynamics model propagates the state using fourth-order Runge-Kutta numerical integration only taking into account the two body gravity and J_2 of the asteroid. The filter measurement models are consistent with the reference truth as shown in Eqns. (61) and (62). In each simulation, the initial $1-\sigma$ formal uncertainties provided to the filter in position and velocity in each axis are 1 km and 50 mm/s. The exploiting triangular structure (ETS) technique previously developed by the authors is used in the UKF time update to reduce computation time with no loss of accuracy by reusing sigma point propagations when possible.⁴

The described filter is implemented with CM, ASNC, and ADMC. Given that a UKF is used, any terms in the process noise algorithms containing Φ or **H** are instead computed using sigma points and the nonlinear dynamics and measurement models. Each of the process noise techniques is utilized with a sliding window of 30 time steps, which is a 2.5 hour interval. Since the filter initially converges very rapidly, adaptation of the process noise covariance is delayed by ten time steps to minimize the violation of the steady state assumption in each of the process noise algorithms. The weighting matrix for ASNC and ADMC is approximated as diagonal using Eqn. (41), which allows for the computationally efficient estimate of $\tilde{\mathbf{Q}}$ shown in Eqn. (49). For ASNC, the model of the DT process noise covariance is approximated using Eqn. (46). For ADMC, α is set to 0.02, and β is a diagonal matrix with each element on the diagonal equal to 1×10^{-5} s⁻¹. When using ADMC, the state in Eqn. (64) is augmented with a set of three empirical accelerations for each spacecraft.

Results

Filter convergence plots of the x-component of the chief position and velocity vectors are provided in Figure 3. For each process noise technique, the convergence behavior is similar in all three axes and for each spacecraft. Fluctuations in the formal uncertainty are largely due to the timevarying number of visible OpNav landmarks. As can be seen in Figure 3 (a), the inherently biased estimate of the DT process noise covariance provided by CM causes filter inconsistency in the estimated velocity vector. This inconsistency leads to occasional spikes in position error. Conversely, ASNC and ADMC maintain filter consistency for both position and velocity. These results highlight the robustness of the proposed algorithms to large and highly correlated process noise.

 Table 3: Case study II mean 3D error computed over the last

 two orbits and averaged over 1000 Monte Carlo simulations.

	СМ	ASNC	ADMC
Position (m)	9.14	4.82	3.89
Velocity (mm/s)	19.1	9.41	8.59

The mean 3D error computed over the last two orbits and averaged over 1000 Monte Carlo simulations is listed in Table 3 for each of the considered process noise techniques. ASNC provided 47% less error in position and 51% less error in velocity than CM. ADMC yielded a reduction of 19% in position error and 9% in velocity error over ASNC. Tracking of the unmodeled accelerations for ADMC is shown in Figure 6 in the appendix. Although further simulations show that CM is able to provide consistent estimation when the process noise is smaller than that of this case study, ASNC consistently provides orbit determination that is at least as accurate and typically 10-30% more accurate in position than CM. Conversely, ADMC only provides an improvement over CM and ASNC if the unmodeled accelerations are accurately estimated. This occurs when the measurement uncertainty is small in comparison to the magnitude of the unmodeled accelerations, and the measurement rate is fast enough to capture the dominant frequency of the unmodeled accelerations. In additional simulations where the unmodeled accelerations were very weakly observable, ADMC often overestimated \widetilde{Q} . In these cases the filter remained consistent, but the navigation accuracy was worse than when using CM or ASNC. Investigating this behavior is a topic of future research.



Figure 3: Filter convergence of a single Monte Carlo run for each considered process noise technique over the first two orbits of simulation. Black lines show the estimation error of the x-component of the chief position (top) and velocity (bottom) vectors. The corresponding formal $3-\sigma$ bound is given by the red region.

Filter computation time is dominated by propagating the 2n + 1 sigma points in the time update where *n* is the number of state variables. For the state shown in Eqn. (64), there are 25 sigma points. The number of sigma points increases to 37 when the state is augmented with empirical accelerations. Since the estimated state comprises two spacecraft states, a traditional UKF would propagate 2(2n + 1) spacecraft orbits over the measurement update interval at each filter call. However, the ETS technique reduces the number of orbit propagations from 50 to 38 for the state shown in Eqn. (64) and from 74 to 56 when the state is augmented with empirical accelerations. As a result, ETS reduces filter run time by approximately 24% regardless of which process noise technique is used. The average run time per filter call for the ETS-UKF when using a fixed process noise covariance was 66 ms for a MATLAB implementation on a 4 GHz Intel Core i7-6700 processor. CM and ASNC only incur an additional 3.4×10^{-3} % and 1.0×10^{-2} % in run time respectively. ADMC increases computation time by about 47% due to the increased number of sigma point propagations.

CONCLUSIONS

This paper presents two new techniques to accurately estimate Kalman filter process noise online for robust orbit determination in the presence of dynamics model uncertainties. The limitations of common orbit determination process noise techniques such as state noise compensation (SNC), dynamic model compensation (DMC), and covariance matching (CM) adaptive filtering are overcome by optimally fusing SNC and DMC with CM. This yields two new techniques called adaptive SNC (ASNC) and adaptive DMC (ADMC). The adaptability of the developed algorithms is a significant advantage over SNC and DMC, which require onerous offline tuning and are not applicable if the process noise varies significantly or the dynamical environment is not well known a priori. The new techniques also provide improved accuracy and robustness over CM. This improvement occurs because the developed algorithms guarantee the discrete time (DT) process noise covariance is positive semi-definite (PSD) without truncating terms, and the DT process noise covariance is constrained according to the underlying continuous time (CT) dynamical model. Another advantage over CM is that the developed algorithms are able to accurately extrapolate the DT process noise covariance over measurement outages. Furthermore, ADMC is well suited for systems with highly colored process noise. These benefits come at relatively little additional computational cost. ASNC is less computationally expensive and simpler to implement than ADMC. On the other hand, ADMC provides superior estimation when the measurement rate and accuracy are sufficient to enable accurate tracking of the unmodeled accelerations. The proposed algorithms have the potential to improve orbit determination for Earth missions and are enabling technologies for asteroid missions where the dynamical environment is poorly known a priori, the process noise is time-varying, and there is limited human intervention.

Future work for both ASNC and ADMC will include estimating the full CT process noise covariance through weighted least squares where the CT process noise covariance is constrained to be PSD. This is a convex program that can be efficiently solved using interior point methods. Estimating the full CT process noise covariance instead of just the main diagonal is expected to increase estimation accuracy in exchange for a higher computational cost. ASNC and ADMC will also be applied to orbital element states, and fusing SNC and DMC with other adaptive filtering techniques will be explored. For ADMC, future research will also include eliminating the need for the tunable forgetting factor. This can likely be accomplished by taking into account how a change in the CT process noise covariance influences the spacecraft state formal covariance through the time update at the following filter call. Furthermore, it should be investigated if ADMC can be improved by estimating the empirical acceleration time correlation constants as part of the filter state and using higher order empirical acceleration dynamical models such as a second order Gauss-Markov process.

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APPENDIX: ADDITIONAL FILTER PERFORMANCE PLOTS



Figure 4: Case study I initial filter convergence for each considered process noise technique over the first 60 seconds of simulation for a single representative filter run. Solid lines show the true error with the shaded region of same color representing the corresponding formal 3- σ bound. For SNC, DMC, ASNC, and ADMC the filter is initialized with a much larger than optimal value of \tilde{Q} . ASNC and ADMC converge to a near optimal value of \tilde{Q} .



Figure 5: Case study I estimation of the unmodeled acceleration for DMC and ADMC for varying initial values of \tilde{Q} for a single representative filter run. In case study I, $\tilde{Q}_0 = 0.206 \text{ m/s}^{5/3}$ is optimal for DMC.



Figure 6: Case study II ADMC tracking of the x-component of the chief unmodeled accelerations over the last two orbits of a single Monte Carlo simulation. Performance was similar in all three axes and for each spacecraft.

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